The Search for Quasi-Periodicity in Islamic 5-fold Ornament

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Introduction

The Penrose tilings are remarkable in that they are non-periodic (have no translational symmetry) but are clearly organised. Their structure, called quasi-periodicity, can be described in several ways, including via self-similar subdivision, tiles with matching rules, and projection of a slice of a cubic lattice in $\mathbb{R}^5$. The tilings are also unusual for their many centres of local 5-fold and 10-fold rotational symmetry, features shared by some Islamic geometric patterns. This resemblance has prompted comparison, and has led some to see precursors of the Penrose tilings and even evidence of quasi-periodicity in traditional Islamic designs. Bonner [2] identified three styles of self-similarity; Makovicky [20] was inspired to develop new variants of the Penrose tiles and later, with colleagues [24], overlaid Penrose-type tilings on traditional Moorish designs; more recently, Lu and Steinhardt [17] observed the use of subdivision in traditional Islamic design systems and overlaid Penrose kites and darts on Iranian designs. The latter article received widespread exposure in the world's press, although some of the coverage overstated and misrepresented the actual findings.

The desire to search for examples of quasi-periodicity in traditional Islamic patterns is understandable, but we must take care not to project modern motivations and abstractions into the past. An intuitive knowledge of group theory is sometimes attributed to any culture that has produced repeating patterns displaying a wide range of symmetry types, even though they had no abstract notion of a group. There are two fallacies to avoid:

- abstraction: *P* knew about *X* and *X* is an example of *Y* therefore *P* knew *Y*.
- deduction: *P* knew *X* and *X* implies *Y* therefore *P* knew *Y*.

In both cases, it is likely that *P* never thought of *Y* at all, and even if he had, he need not have connected it with *X*.

In this article I shall describe a tiling-based method for constructing Islamic geometric designs. With skill and ingenuity, the basic technique can be varied and elaborated in many ways, leading to a wide variety of complex and intricate designs. I shall also examine some traditional designs that exhibit features comparable with quasi-periodic tilings, use the underlying geometry to highlight similarities and differences, and assess the evidence for the presence of quasi-periodicity in Islamic art.

A few comments on terminology. Many of the constructions are based on tilings of the plane. A *patch* is a subset of a tiling that contains a finite number of tiles and is homeomorphic to a disc. I use *repeat unit* as a generic term for a template that is repeated using isometries to create a pattern; it is not so specific as period parallelogram or fundamental domain. A design or tiling with *radial symmetry* has a single centre of finite rotational symmetry. The other terminology follows [8] for tilings, supplemented by [33] for substitution tilings.

Islamic Methods of Construction

Although the principles of Islamic geometric design are not complicated, they are not well-known. Trying to recover the principles from finished artwork is difficult, as the most conspicuous elements in a design are often not the compositional elements used by the designer. Fortunately medieval documents that reveal some of the trade secrets have survived. The best of these documents is manuscript scroll MS.H.1956 in the library of the Topkapi Palace, Istanbul. The scroll itself is a series of geometric figures drawn on individual pages, glued end to end to form a continuous sheet about 33 cm high and almost 30 m long. It is not a 'how to' manual, as there is no text, but it is more than a pattern book as it shows construction lines. A half-size colour reproduction can be found in [25], which also includes annotations to show the construction lines and marks scored into the paper with a stylus, which are not visible in the photographs. References in this article to numbered panels of the Topkapi Scroll use the numbering in [25].

Islamic designs often include star motifs. These come in a variety of forms but, in this article, we need only a few simple shapes that correspond to the regular star polygons of plane geometry. Taking n points equally spaced around a circle and connecting points d intervals apart by straight lines produces the star polygon denoted by $\{n/d\}$. This, however, is the star of the mathematician; it is rare for an artist to use the whole figure as an ornamental motif. More often, the middle segments of the sides are discarded.

Many of the early Islamic designs are created by arranging 6-, 8- or 12-point stars at the vertices of the standard grids of squares or equilateral triangles. The more general rhombic lattice allows other stars to be used. An example based on {10/3} is shown in Figure 1(a). The angles in the rhombus are 72° and 108°, both being multiples of 36°—the angle between adjacent spikes of the star. Draw a set of circles of equal radius centred on the vertices of the lattice and of maximal size so that there are points of tangency. Place copies of the star motif in the circles so that spikes fall on the edges of the lattice. This controls the spacing and orientation of the principal motifs, but the design is not yet complete. There are some spikes of each motif that are not connected to a neighbouring motif but are free and point into the residual spaces between the circles. The lines bounding these free spikes are extended beyond the circumcircle until they meet similarly produced lines from nearby stars. This simple procedure bridges the residual spaces and increases the connectivity of the star motifs. The same pattern of interstitial filling should be applied uniformly to all the residual spaces and the symmetry of the design as a whole should be preserved as far as possible. The result is shown in Figure 1(c). In this case the kites in the interstitial filling are congruent to those in the star. This pattern is one of the most common decagonal designs, and we shall name it the 'stars and kites' pattern for reference.

This basic approach produces a limited range of periodic designs with small repeat units and it only works for stars with an even number of points. A more general method that can be used with all stars, and also enables combinations of different stars to be used in a single design, is based on edge-to-edge tilings containing regular convex polygons with more than four sides. Figure 1(b) shows a tiling formed by packing decagons together, leaving non-convex hexagonal tiles between them. After placing {10/3} stars in each decagon tile, we use the same kind of interstitial filling procedure as before to develop the pattern in the hexagons.

This change from circle to polygon may seem minor, but it gives rise to a range of generalisations. We are no longer restricted to a lattice arrangement of the stars—any tiling will suffice. The tiling may contain regular polygons of different kinds allowing different star motifs to be combined in the same design; the tiling naturally determines the relative sizes of the different stars. We can even discard the regular star motifs that initiate the interstitial filling and seed the pattern generation process from the tiling itself. In this last case, we place a pair of short lines in an X configuration at the midpoint of each edge, then extend them until they encounter other such lines—this is similar to applying interstitial filling to every tile. The angle that the lines make with the edges of the tiling, the *incidence angle*, is a parameter to be set by the artist and it usually takes the same value at all edges. There is no requirement to terminate the line extensions at the first point of intersection; if there are still large empty regions in the design, or it is otherwise unattractive, the lines can be continued until new intersections arise.

This technique, known as 'polygons in contact' (PIC), was first described in the West by Hankin [9–13], who observed the polygonal networks scratched into the plaster of some designs, while working in India. Many panels in the Topkapi Scroll also show a design superimposed on its underlying polygonal network. Although the purpose of the networks is not documented, it does not seem unreasonable to interpret them as construction lines. Bonner [2, 3] argues that PIC is the only system for which there is evidence of historical use by designers throughout the Islamic world. The method is versatile and can account for a wide range of traditional patterns, but it is not universally applicable. An alternative approach is used by Castéra [5],

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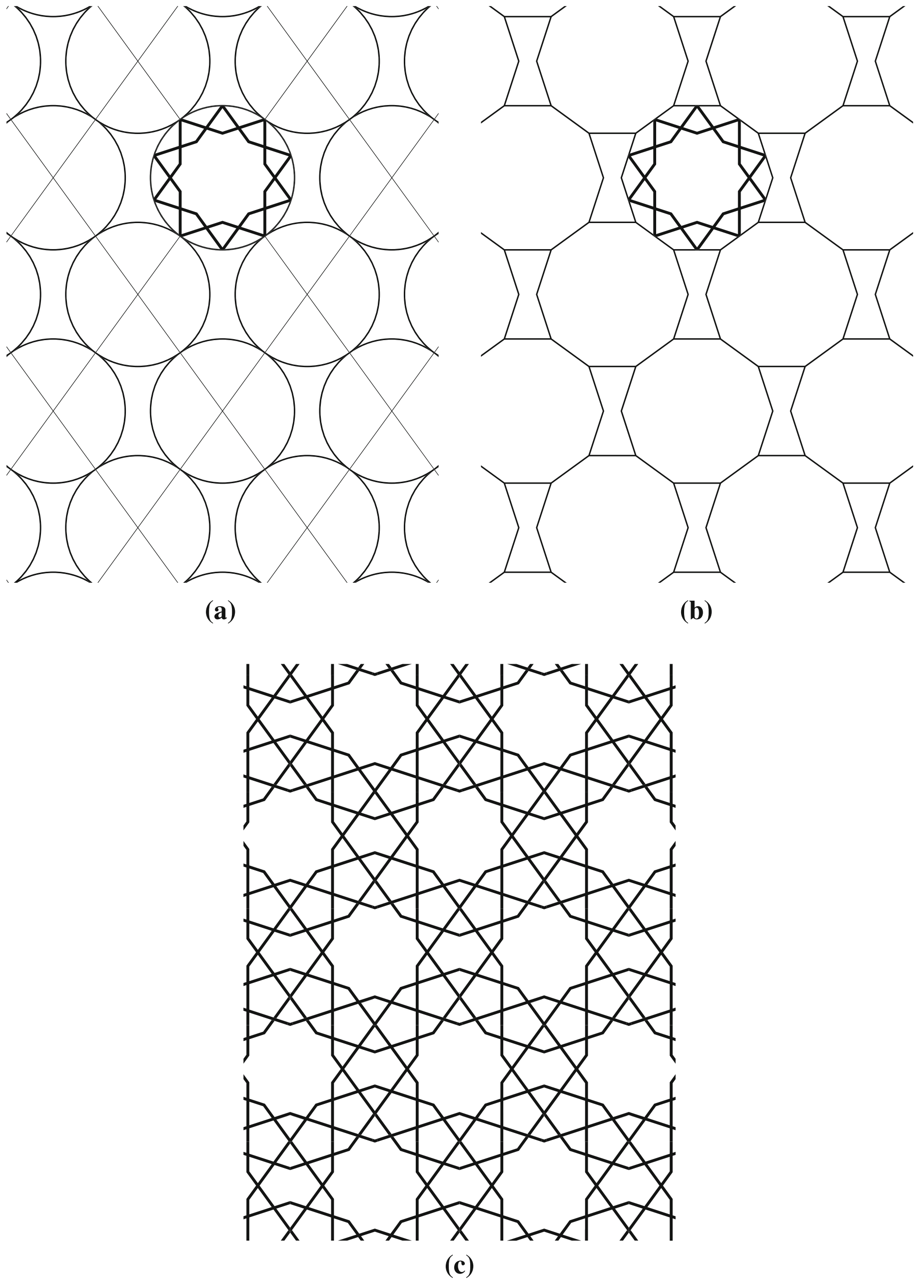
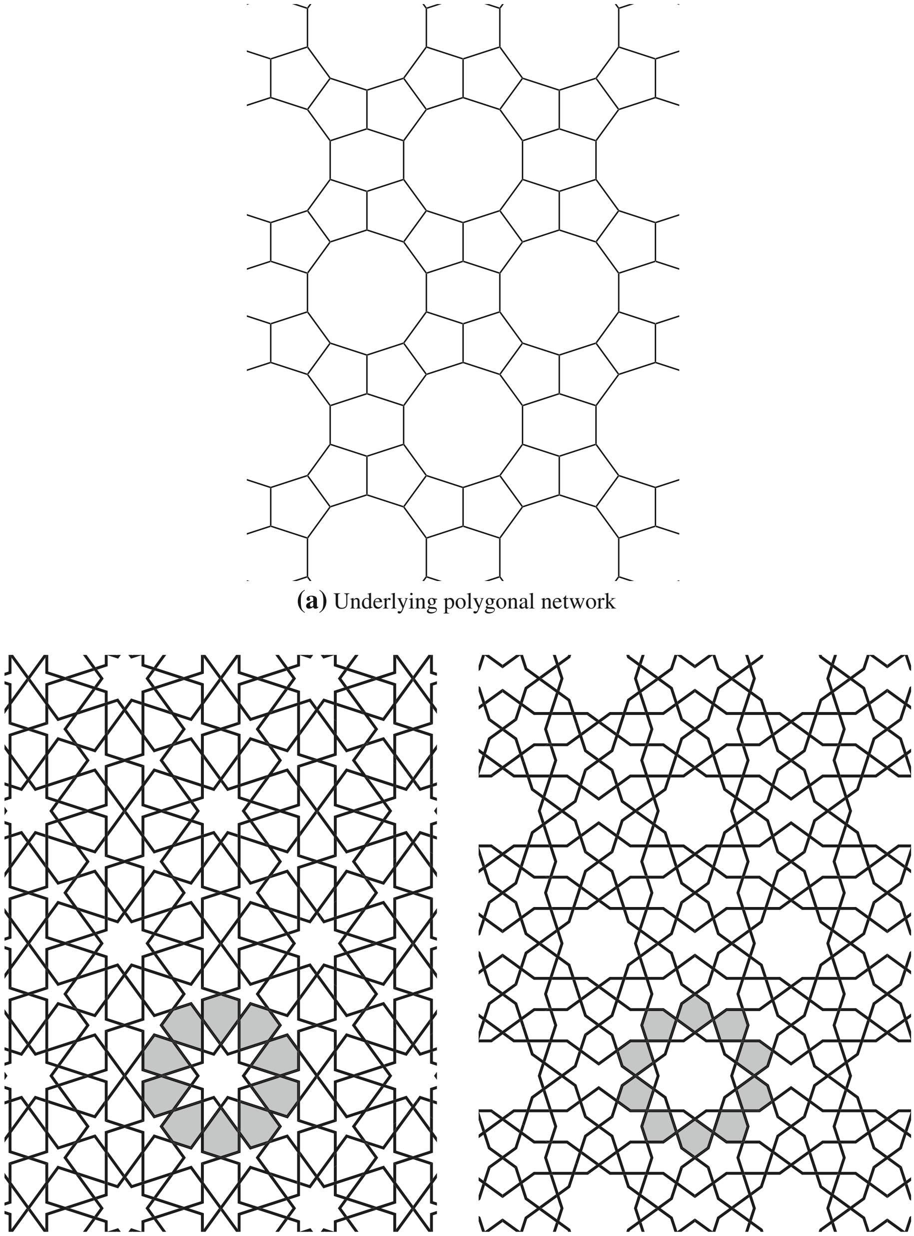


Figure 1. The 'stars and kites' pattern.

who arranges the shapes seen in the final design without using a hidden grid.

The PIC method is illustrated in the next four figures. Figure 2 shows two designs produced from a tiling by regular decagons, regular pentagons, and irregular convex hexagons. In part (b), a star motif based on {10/4} is placed in the decagon tiles, which gives an incidence angle of 72° for the other edges; the completed design is one of the most widespread and frequently used of all star patterns. Part (c) shows a design that is common in Central Asia and based on {10/3} with an incidence angle of 54°. A {10/2} star and an incidence angle of 36° reproduces the stars and kites pattern. The design in Figure 3 is from [14] and contains star motifs based on {7/3}; in the tiling the 7-gons are regular but the pentagons are not. Figure 4 is based on a tiling containing regular 9-gons and 12-gons. I have chosen an incidence angle of 55° to make the convex 12-gon elements in the design into regular polygons and some line segments inside the non-convex hexagonal tiles join up without creating a corner, but, as a consequence, neither star motif is geometrically regular. Plates 120–122 in [4] are traditional designs based on the same tiling. Figure 5 shows a design with 10-fold rotational symmetry based on panel 90a of the Topkapi Scroll, which Necipoğlu labels as a design for a dome [25]. The original panel shows a template for the figure containing one-tenth of the pattern with the design in solid black lines superimposed on the tiling drawn in red dotted lines. Notice that some of the tiles are two-tenths and three-tenths sectors of a decagon. Domes were also decorated by applying PIC to polyhedral



(a) Underlying polygonal network

(b) Incidence angle 72°

(c) Incidence angle 54°

Figure 2. A tiling and two star patterns derived from it. The petals of a rose motif in each pattern are highlighted.

networks. Patterns with a lower concentration of stars were produced by applying PIC to *k*-uniform tilings composed of regular 3-, 4-, 6-, and 12-sided polygons—see plates 77, 97, and 142 in [4] for some unusual examples.

The two designs of Figure 2 display another common Islamic motif. In each design, a set of hexagons surrounding a star has been highlighted in grey. The enlarged star motif is called a *rose* and the additional hexagons are its *petals*. In this case, the rose arises because the decagon in the underlying tiling is surrounded by equilateral polygons, but they can also be constructed using a set of tangent circles around the circumcircle of the star [16] and used as compositional elements in their own right.

You can see the PIC method in action and design your own star patterns using Kaplan's online Java applet [34]—you select a tiling and the incidence angles of the star motifs, then inference logic supplies the interstitial pattern.

The tilings used as the underlying networks for the PIC method of construction often have a high degree of symmetry, and they induce orderly designs. Islamic artists also produced designs that appear to have a more chaotic arrangement of elements with local order on a small scale but little long-range structure visible in the piece shown. Panels in the Topkapi Scroll reveal that these designs, too, have an underlying polygonal network assembled from copies of a small set of equilateral tiles (see Figure 6) whose angles are multiples of 36°:

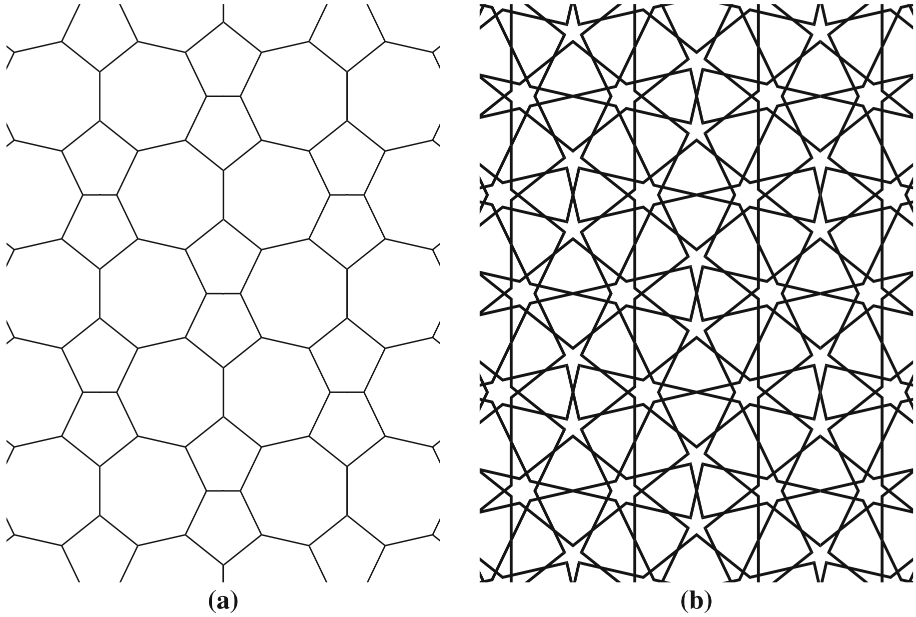


Figure 3. Design containing regular 7-point stars.

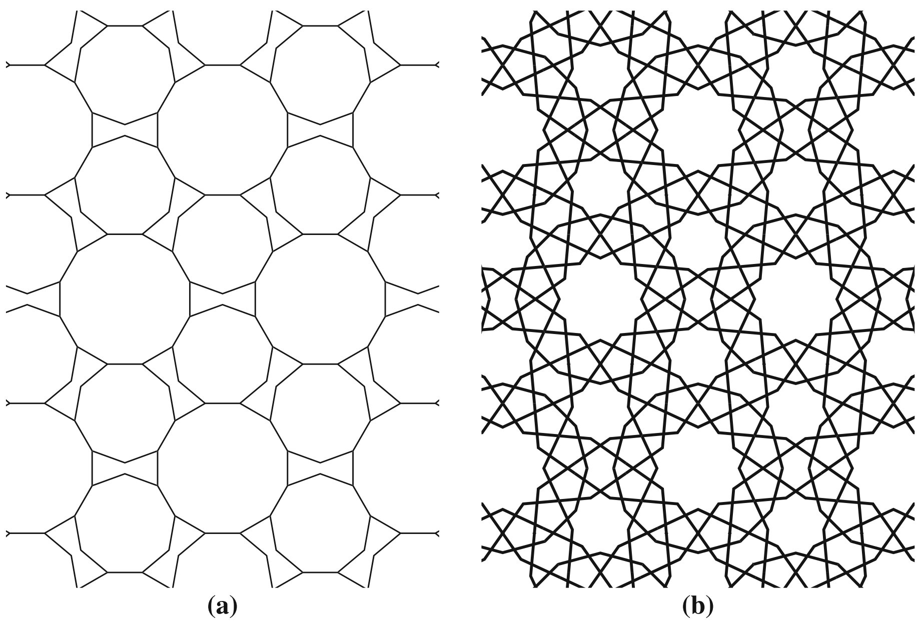


Figure 4. Design containing 9- and 12-point stars.

- a rhombus with angles 72° and 108°
- a regular pentagon (angles 108°)
- a convex hexagon with angles 72° and 144°—the bobbin
- a convex hexagon with angles 108° and 144°—the barrel
- a non-convex hexagon with angles 72° and 216°—the bow-tie
- a convex octagon with angles 108° and 144°
- a regular decagon (angles 144°).

The motifs on the tiles are generated using the PIC method with an incidence angle of 54°. The barrel hexagon and the decagon have two forms of decoration. One decagon motif is just the star {10/3} and its constituent kites are congruent to those on the bow-tie; the other decagon motif is more complex and the symmetry is reduced from 10-fold to 5-fold rotation.

The shapes of the tiles arise naturally when one tries to tile with decagons and pentagons. The bow-tie and barrel hexagons are familiar from the previous figures. The octagon and the remaining hexagon are shapes that can be obtained as the intersection of two overlapping decagons. The motif on the hexagon resembles a spindle or bobbin wound with yarn. This distinctive motif is easy to locate in a

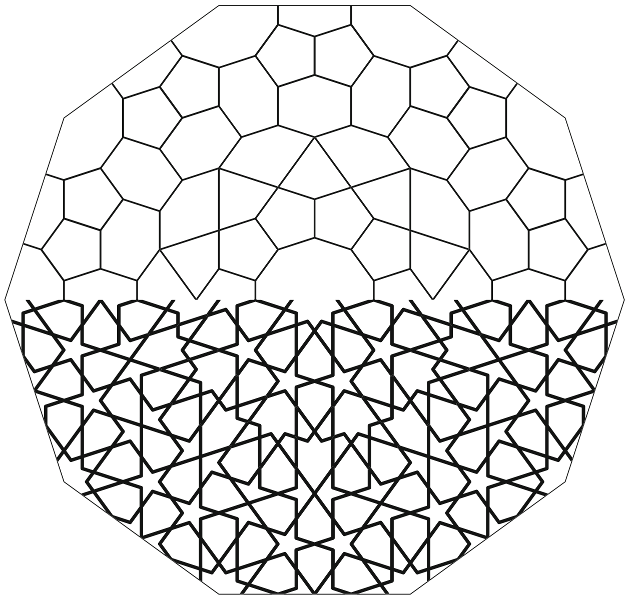


Figure 5. Design from panel 90a of the Topkapi Scroll.

design, and its presence is a good indication that the design could be constructed from these tiles.

The promotion of irregular tiles from supplementary shapes to compositional elements in their own right marked a significant development in Islamic design. Regarding the tiles as the pieces of a jigsaw allows a less formal approach to composition. A design can be grown organically in an unplanned manner by continually attaching tiles to the boundary of a patch with a free choice among the possible extensions at each step. This new approach gave artists freedom and flexibility to assemble the tiles in novel ways and led to a new category of designs. It seems to have been a Seljuk innovation as examples started to appear in Turkey and Iran in the 12th–13th centuries. The widespread and consistent use of these decorated tiles as a design system was recognised by Lu and Steinhardt [17]; similar remarks appear in Bonner [2] and the tiles are also used by Hankin [10].

Figure 7 shows small patches of tiles. There are often multiple solutions to fill a given area. Even in the simple combination of a bobbin with a bow-tie shown in part (a), the positions of the tiles can be reflected in a vertical line so that the bow-tie sits top-right instead of top-left. The patch in part (d) can replace any decagonal tile with a consequent loss of symmetry, as the bow-tie can point in any of ten directions. Patches (b) and (c) are another pair of interchangeable fillings with a difference in symmetry.

Figure 8 shows some traditional designs made from the tiles. Parts (a) and (b) are from panels 50 and 62 of the Topkapi Scroll, respectively; in both cases the original panel shows a template with the design in solid black lines superimposed on the tiling drawn in red dots. The designs in parts (b), (c), and (d) are plates 173, 176, and 178 of [4]. The designs of (e) and (f) are Figures 33 and 34 from [16]. The edges of the tilings are included in the figures to show the underlying structure of the designs, but in the finished product these construction lines would be erased to leave only the interlaced ribbons. This conceals the underlying framework and helps to protect the artist's method. The viewer sees the polygons of the background outlined by the ribbons, but these are artifacts of the construction, not the principal motifs used for composition.

The internal angles in the corners of the tiles are all multiples of 36° so all the edges in a tiling will point in one of five directions—they will all lie parallel to the sides of a pentagon. Fitting the tiles together spontaneously produces regular pentagons in the background of the interlacing, and centres of local 5-fold or 10-fold rotational symmetry in the design. This symmetry can be seen in some of the configurations of Figure 7. However, in patterns generated by translation of a template, this symmetry must break down and cannot hold for the design as a whole. This is a consequence of the crystallographic restriction: the rotation centres in a periodic pattern can only be 2-, 3-, 4- or 6-fold. This was not proved rigorously until the 19th century but it must surely have been understood on an intuitive level by the Islamic pattern makers. Perhaps these tilings were appealing precisely because they contain so many forbidden centres; they give the illusion that one can break free from this law of nature. Unfortunately, when a large enough section of a tiling is shown for the periodicity to be apparent, any (global) rotation centres are only 2-fold, and

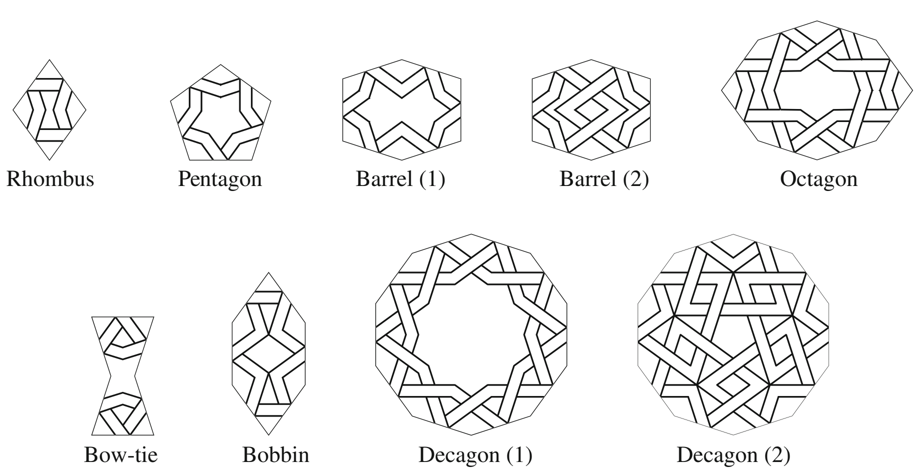


Figure 6. An Islamic set of prototiles.

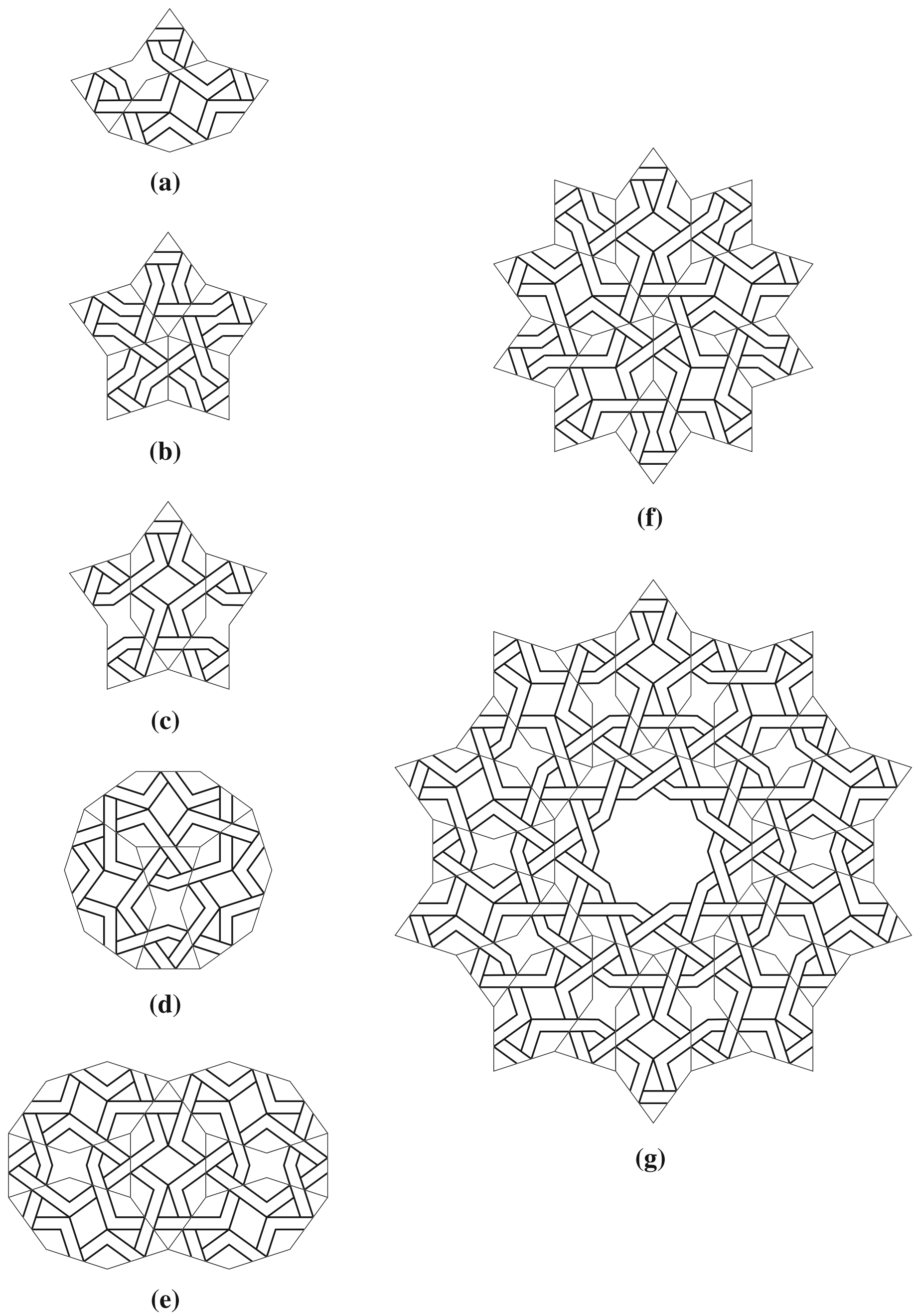


Figure 7. Small patches of tiles.

the symmetry type of the (undecorated) tiling is usually one of *pgg*, *pmm* or, more commonly, *cmm*.

Figure 9(b) shows the design on one wall of the Gunbad-i Kabud (Blue Tower) in Maragha, north-west Iran; similar designs decorate the other sides of the tower. At first sight the design appears to lack an overall organising principle but it fits easily into the framework shown in Figure 9(a). Centred at the bottom-right corner of the panel is the patch of Figure 7(g) surrounded by a ring of decagons. A similar arrangement placed at the top-left corner abuts the first, leaving star-shaped gaps. The rings of decagons are filled with the patch of Figure 7(d) with the bow-ties facing outwards, except for the one on the bottom edge of the panel, which is filled with a decagonal tile. The star-shaped gaps are filled with the five rhombi of Figure 7(b). The design does contain irregularities and deviations from this basic plan, particularly in the bottom-left corner of the panel. Also the decagon in the top-left corner is filled with Figure 7(d) rather than a decagonal tile.

Figure 9(a) can also be taken as the foundation of the design shown in Figure 10. The centres of the rose motifs in the centre of the figure and in the top-left corner are diagonally opposite corners of a rectangle that is a repeat unit for the design. The underlying framework in this rectangle is the same as that of the Maragha panel. The full design is generated from this cell by reflection in the sides of the rectangle. Note that it is the *arrangement* of the tiles that is

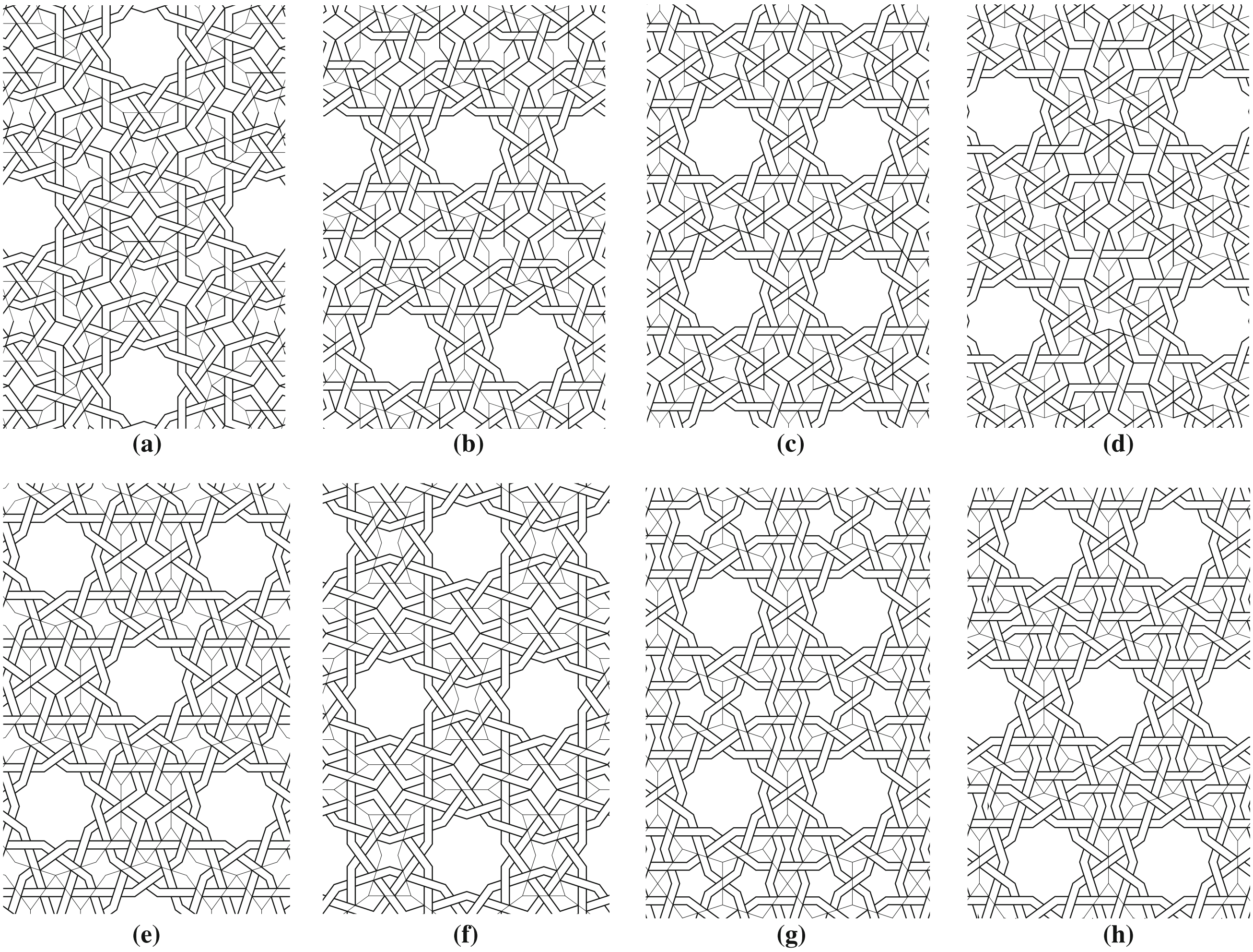


Figure 8. Periodic designs.

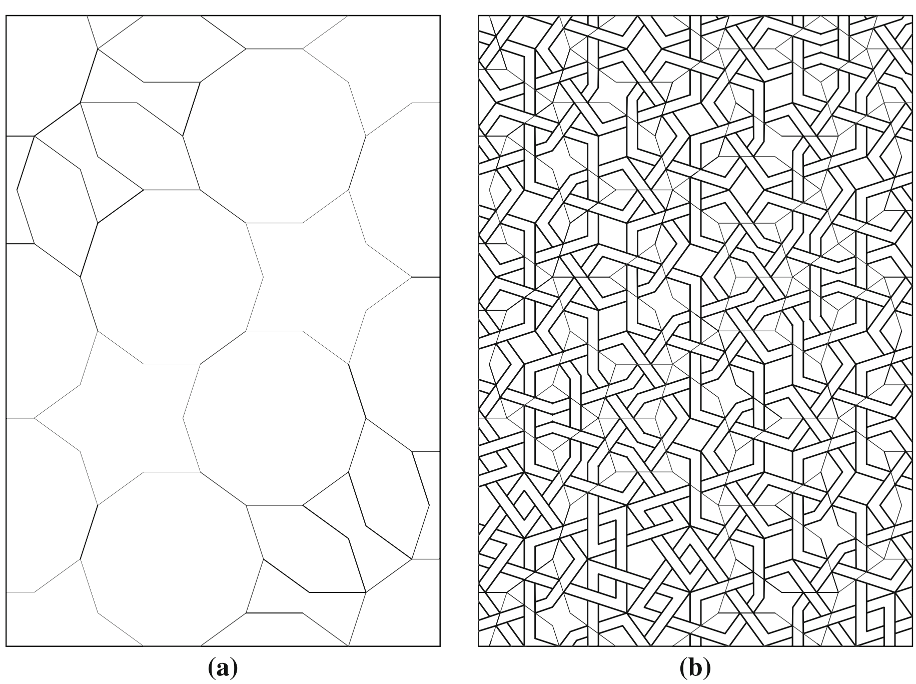


Figure 9. Design from the Gunbad-i Kabud, Maragha, Iran.

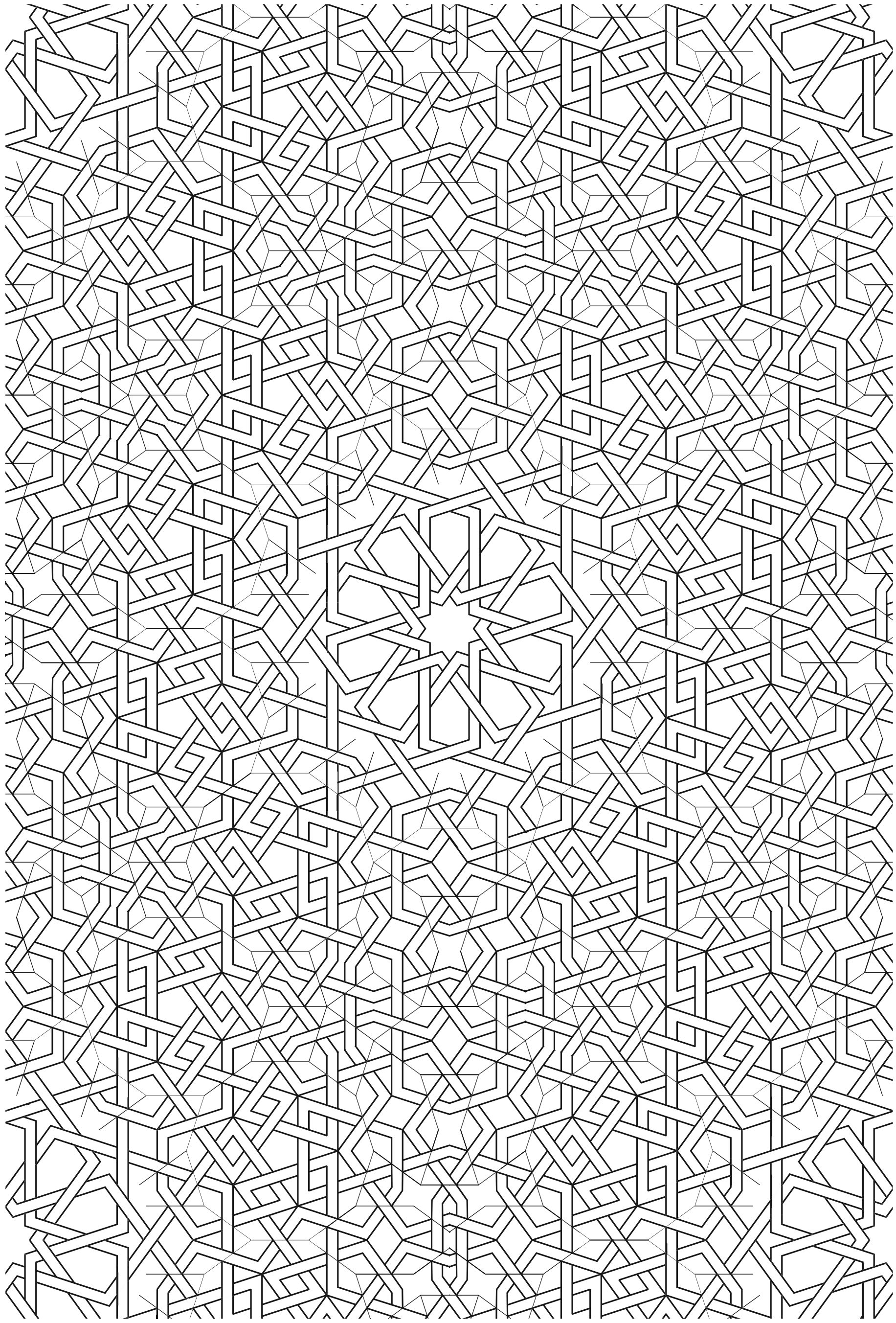


Figure 10. Design from the Karatay Madrasa, Konya, Turkey.

reflected, not the tiles with their decorative motifs; the interlacing of the full design remains alternating. The boundaries of the unit rectangle are mostly covered by the sides of tiles or mirror lines of tiles, both of which ensure continuity of the tiling across the joins. However, in the top-right and bottom-left corners (the cell has 2-fold rotational symmetry about its centre), the tiles do not fit in the rectangle but overhang the edges. This is not a problem with this method of generating designs: the overhanging tiles are simply cut to fit and the reflections take care of the continuity of the ribbons. In Figure 10 this is most obvious in the middle near the bottom where pairs of bow-ties and bobbins merge. The centre of the tiling can be filled with the patch shown in Figure 7(g) but this has been discarded in favour of a large rose motif. A different construction for this pattern is presented by Rigby in [26].

When experimenting with the tiles of Figure 6, one soon learns that those in the top row are more awkward to use than the others—the 108° angles must occur in pairs around a vertex and this limits the options. Indeed many designs avoid these tiles altogether and are based solely on the three shapes in the bottom row. The design in Figure 11 is unusual in that it is largely composed of awkward tiles (rhombi, pentagons, and octagons) together with a few bobbins. The large star-shaped regions in the tiling can be filled with the patch shown in Figure 7(f),

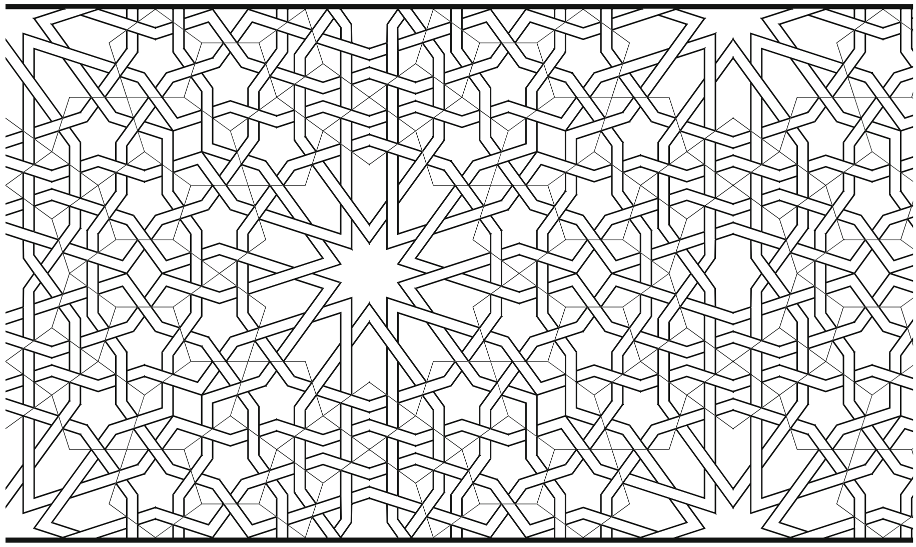
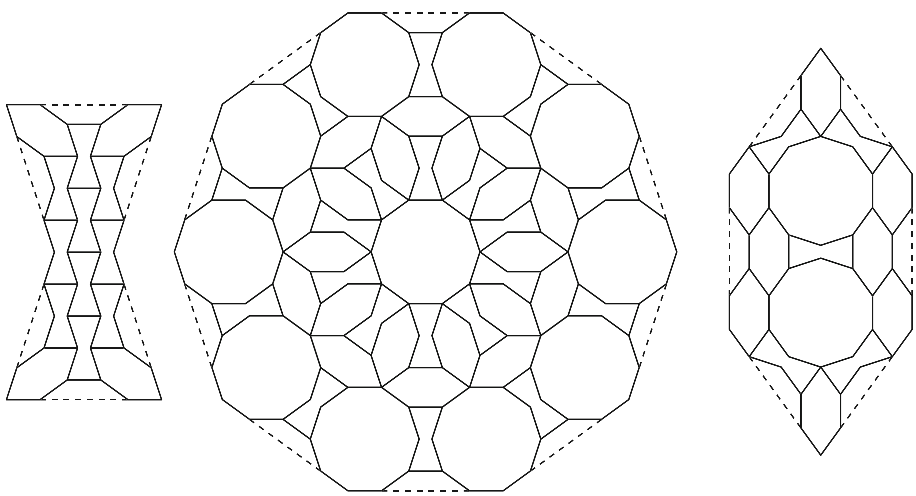


Figure 11. Design from the Sultan Han, Kayseri, Turkey.



(a) (b) (c)

Figure 12. Subdivisions of three tiles into smaller copies of the same three tiles. The scale factor is $\frac{1}{2}(7+\sqrt{5}) \approx 4.618$.

continuing the use of the same set of tiles, but instead this motif is replaced by the star {10/4}.

Once a design has been constructed, it can be finished in different ways according to context and the materials used. In some of the accompanying figures, the regions have been given a proper 2-colouring (chessboard shading), in others the lines have been made into interlaced ribbons. The basic line drawing can also be used by itself as when it is inscribed in plaster.

What is Quasi-Periodicity?

The discovery of crystalline metal alloys with 5-fold symmetry in their diffraction patterns caused great excitement in the 1980s. Sharp spots in a diffraction pattern are evidence of long-range order which, at that time, was synonymous with periodicity, but 5-fold rotations are incompatible with the crystallographic restriction so a new kind of phenomenon had been observed. The novel solids became known as quasi-crystals and the underlying order as quasi-periodicity. For crystallographers, the production of sharply defined points in a diffraction pattern is a defining characteristic of quasi-periodicity. In the study of the decorative arts, however, the term 'quasi-periodic' is used somewhat informally and does not have an agreed definition. Readers should be aware of this potential source of confusion when comparing papers. For the tilings and the related geometric designs discussed in this article, one option is to impose a homogeneity condition on the distribution of local configurations of tiles (this is weaker than the crystallographic definition). This and other properties will be illustrated through the following example.

The example is constructed from the patches shown in Figure 12. The patches are chosen only to demonstrate the technique and not for any artistic merit—the unbalanced distribution of bow-ties leads to poor designs. Any patch tiled by bow-ties, bobbins, and decagons can be converted

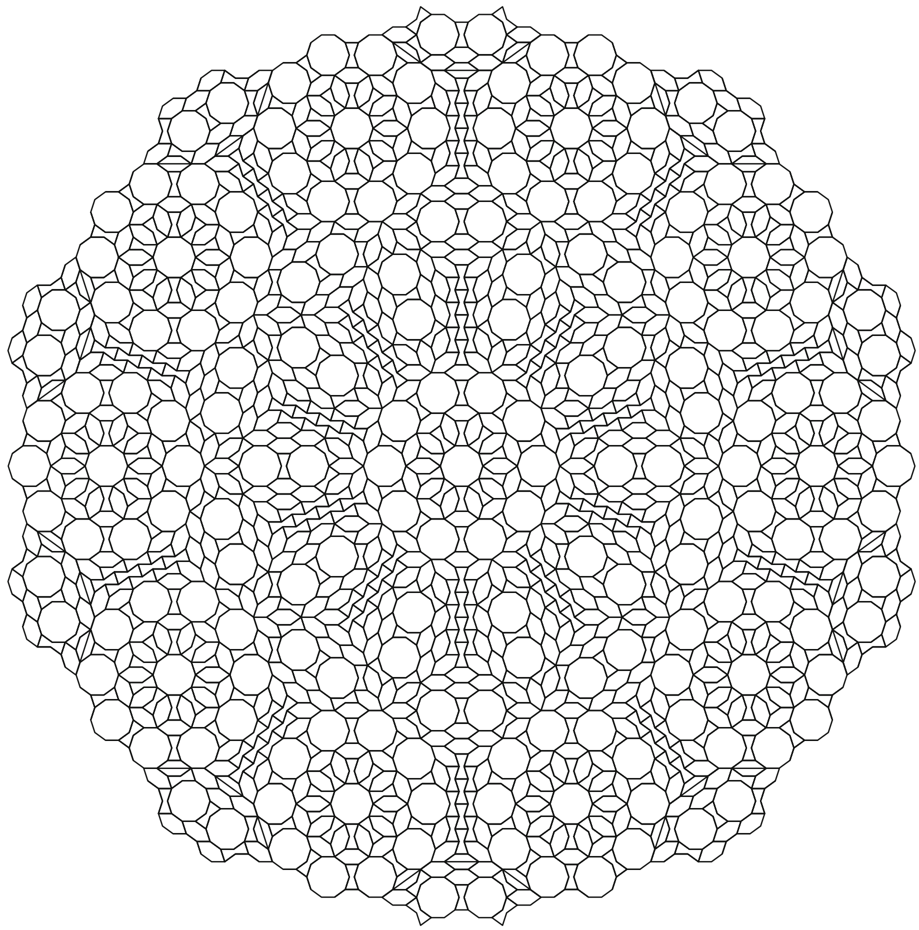


Figure 13. A step in the construction of a quasi-periodic tiling.

into a larger such patch by subdividing each tile as shown in the figure and then scaling the result to enlarge the small tiles to the size of the originals. This process of 'subdivide and enlarge' is called *inflation*. Each side of each composite tile is formed from two sides of small tiles and the major diagonal of a small bobbin; in the inflated tiling the half-bobbins pair up to form complete tiles.

Let P_0 be a single decagon and let P_{i+1} be the patch obtained by inflating P_i for all $i \in \mathbb{N}$. Figure 12(b) shows P_1 and Figure 13 shows P_2. We can iterate the inflation process to tile arbitrarily large regions of the plane. Furthermore, because P_1 contains a decagon in the centre, P_{i+1} contains a copy of P_i in the middle. Therefore P_{i+1} is an extension of P_i, and by letting i go to infinity we can extend the patch to a tiling, P_∞, of the whole plane. Notice that the symmetry of the initial patch is preserved during inflation so P_∞ will have a global centre of 10-fold symmetry and hence cannot be periodic.

In general, inflation only provides the ability to create arbitrarily large patches that need not be concentric, so some work is required to show that the limit exists and it is a tiling of the plane [19]. Two tilings are said to be *locally indistinguishable* if a copy of any patch from one tiling occurs in the other tiling, and *vice versa*. The family of *substitution tilings* defined by the prototiles and subdivisions shown in Figure 12 is the set of all tilings that are locally indistinguishable from P_∞. There are, in fact, an uncountable number of tilings in the family but any patch in any one of them will be contained in some P_n.

The basic combinatorial properties of a substitution tiling based on a finite set of n prototiles $T_1,...,T_n$ can be encoded in an $n \times n$ matrix: the entry in column j of row i is the number of small T_i in a composite T_j. For the example here with the tiles in the order bow-tie, bobbin, decagon, this substitution matrix is

$$\begin{pmatrix} 10 & 5 & 20 \\ 7 & 11 & 25 \\ 0 & 2 & 11 \end{pmatrix}.$$

A matrix is said to be *primitive* if some power of it has only positive non-zero entries. If a substitution matrix is primitive then the patch of tiles produced by repeated inflation of any tile will eventually contain copies of all the prototiles. Properties of the tiling can be derived from the algebraic properties of a primitive matrix. For example, the largest eigenvalue is the square of the scale factor of the inflation and the corresponding eigenvector contains the relative frequencies of the prototiles in a full tiling of the plane; the corresponding eigenvector of the transposed matrix contains the relative areas of the three prototiles. In our example the frequency eigenvector is $\left(5+5\sqrt{5}, 5+7\sqrt{5}, 4\right)$. Since some of the ratios between the entries are irrational, any substitution tiling made from these subdivisions is non-periodic [30, 31].

Although our substitution tilings have no translational symmetry, they do share some properties with periodic tilings. First each tiling is edge-to-edge; it is constructed from a finite number of shapes of tile, each of which occurs in a finite number of orientations; there are finitely many ways to surround a vertex. The tiling is said to have *finite local complexity*. For primitive substitution tilings this has an important consequence: given any patch X in the tiling there is some number R such that a disc of radius R placed anywhere on the tiling will contain a copy of X. A tiling with this property is called *repetitive*. This means that copies of any finite portion of the tiling can be found evenly distributed throughout the tiling. You cannot determine which part of the tiling is shown in any finite diagram of it.

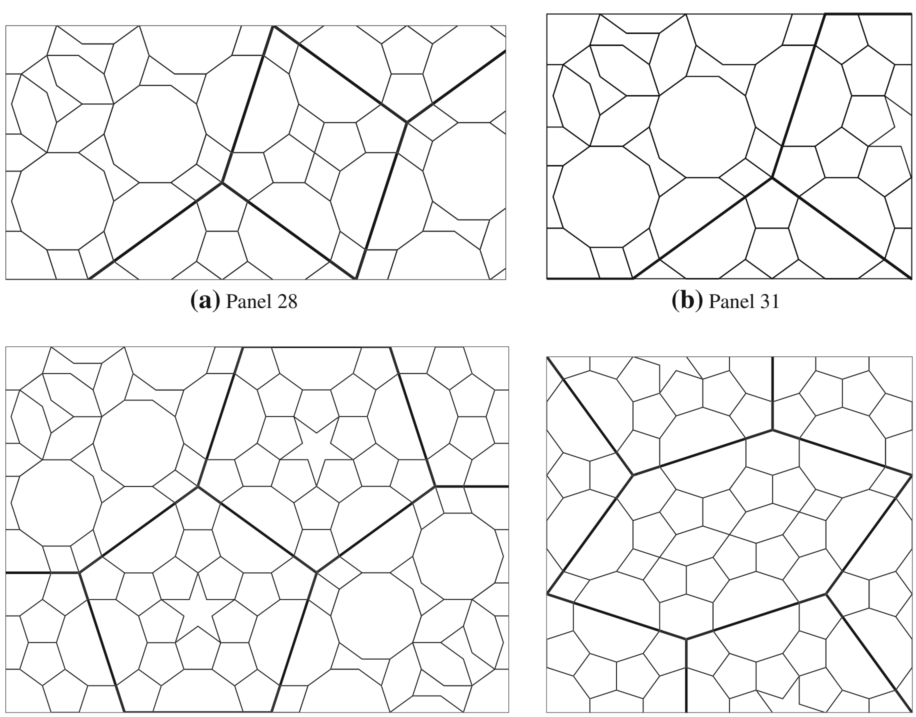
For the purposes of this article, a tiling is called *quasi-periodic* if it is non-periodic, has finite local complexity, and is repetitive. By extension we can call an Islamic design constructed using the PIC method quasi-periodic if its underlying polygonal network is a quasi-periodic tiling. Unfortunately, it is impossible to tell from any finite subset of a tiling whether it is quasi-periodic or not. So, in order to assert that a tiling could be quasi-periodic, we need to identify a process such as inflation that could have been used to generate the piece shown and can also be used to generate a complete quasi-periodic tiling.

Multi-level Designs

Some panels of the Topkapi Scroll show designs of different scales superimposed on one another. This interplay of designs on multiple scales is a feature of some large Islamic designs found on buildings where viewers experience a succession of patterns as they approach. From a distance, large-scale forms with high contrast dominate but, closer in, these become too large to perceive and smaller forms take over. Early methods to achieve this transition from big and bold through medium range to fine and delicate were simple, often just a matter of progressively filling voids in the background to leave a design with no vacant spaces. (There is a secondary pattern of this form on the Gunbad-i Kabud.) Differences in size and level of detail were expressed using variation in density, depth of carving, colour and texture. Later designs are more ambitious and use the same style on more than one scale. It is even possible to re-use the same pattern.

Designs that can be read on several scales are often referred to as *self-similar* but this term itself has multiple levels of meaning. In its strictest sense it means *scale invariant*: there is a similarity transformation (an isometry followed by an enlargement) that maps the design onto itself. The transformation can be weakened to a topological equivalence—for example the homeomorphisms in iterated function systems leading to fractals. In a weaker sense still, it means only that motifs of different scales resemble each other in style or composition but are not replicas. We shall use the term *hierarchical* for multi-level designs of this latter form.

In panel 28 of the Topkapi Scroll three drawings are superimposed on the same figure: a small-scale polygonal network is drawn in red dots, the corresponding small-scale design is drawn in a solid black line, and a large-scale design is added in a solid red line. The polygonal network corresponding to the large-scale design is not shown but can be deduced—the two polygonal networks are shown superimposed in Figure 14(a). The other parts of the figure



(a) Panel 28 **(b)** Panel 31 **(c)** Panel 32 **(d)** Panel 34

Figure 14. Underlying 2-level polygonal networks of panels from the Topkapi Scroll.

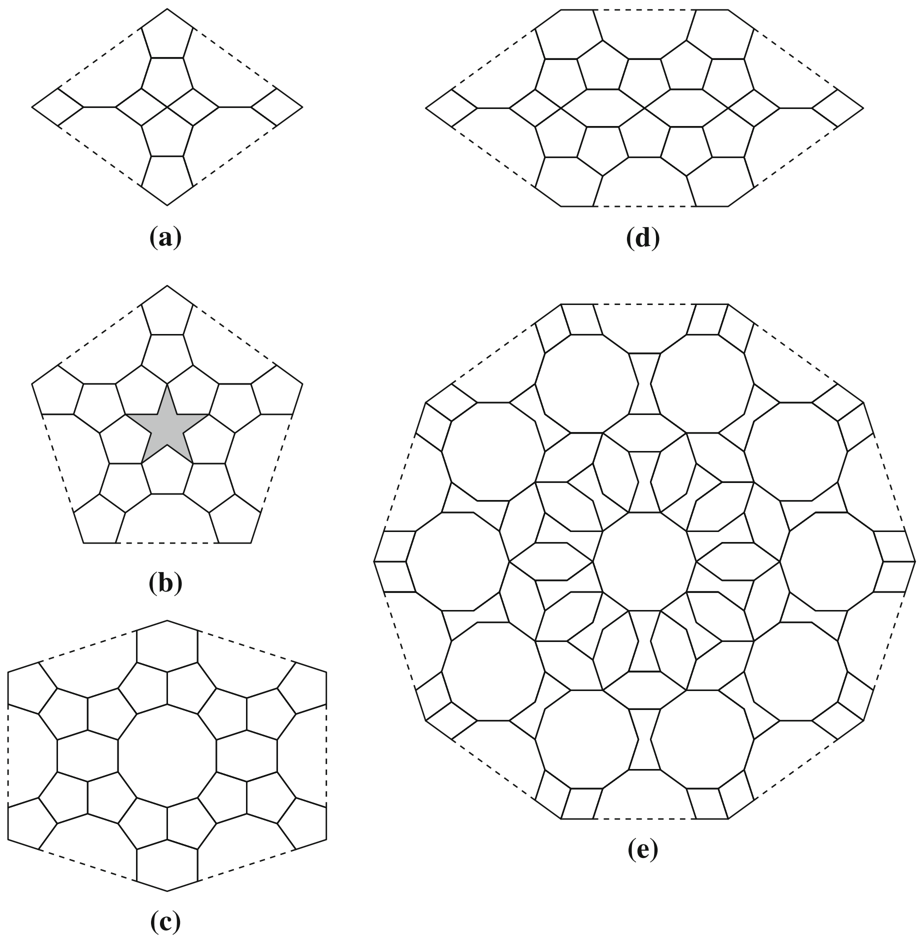


Figure 15. Subdivisions derived from the Topkapi Scroll. The scale factor is $3+\sqrt{5}\approx 5.236$.

show the polygonal networks underlying three more 2-level designs from the scroll, but neither of the networks is shown in these panels, only the finished 2-level designs in black and red.

Superimposing the large- and small-scale polygonal networks of these panels reveals subdivisions of some of the tiles: a rhombus in panel 28, two pentagons in panel 32, and a bobbin in panel 34. In all cases, the side of a composite tile is formed from the sides of two small tiles and the diagonal of a small decagon. We can also identify the fragments of the large-scale polygons cropped by the boundaries of the panels. These panels are not arbitrarily chosen parts of a design—they are templates to be repeated by reflection in the sides of the boundary rectangle. Although a superficial glance at Figure 14(d) might suggest that the large-scale network is a bobbin surrounded by six pentagons, a configuration that can be seen in the small-scale network, reflection in the sides generates rhombi, pentagons, and barrels. The large-scale design generated by panel 31 is shown in Figure 8(g). Panel 28 appears to be truncated on the right and is perhaps limited by the available space. If it had 2-fold rotational symmetry about the centre of the large rhombus, the large-scale design would be that of Figure 8(h). A consistent choice of subdivision emerges in all four panels and the subdivisions of the five tiles used are shown in Figure 15. I believe this has not been reported before.

Figure 16 shows my 2-level design based on panel 32. The composite tiles generate the large-scale design (shown in grey) and the small tiles generate a small-scale design (black and white) that fills its background regions. The barrel tile has two forms of decoration: I have used the simple motif for the large-scale design and the other motif on the small-scale design. Completing the small-scale design in the centre of a composite pentagon is problematic. For a pentagon of this scale, only a partial subdivision is possible: once the half-decagons have been placed, one is forced to put pentagons at the corners; only a pentagon or a barrel can be adjacent to the corner pentagons, and both cases lead to small areas that cannot be tiled. The grey area in Figure 15(b) indicates one such essential hole. I have chosen a slightly different filling from the one in the Topkapi Scroll. The large-scale design is that of Figure 2(c).

Figure 17 gives a similar treatment to panel 34. It contains four copies of the template rectangle shown in Figure 14(d), two direct and two mirror images. In this case, the large-scale pattern is expressed using shading of the regions. Examples of both styles can be found on buildings in Isfahan, Iran.

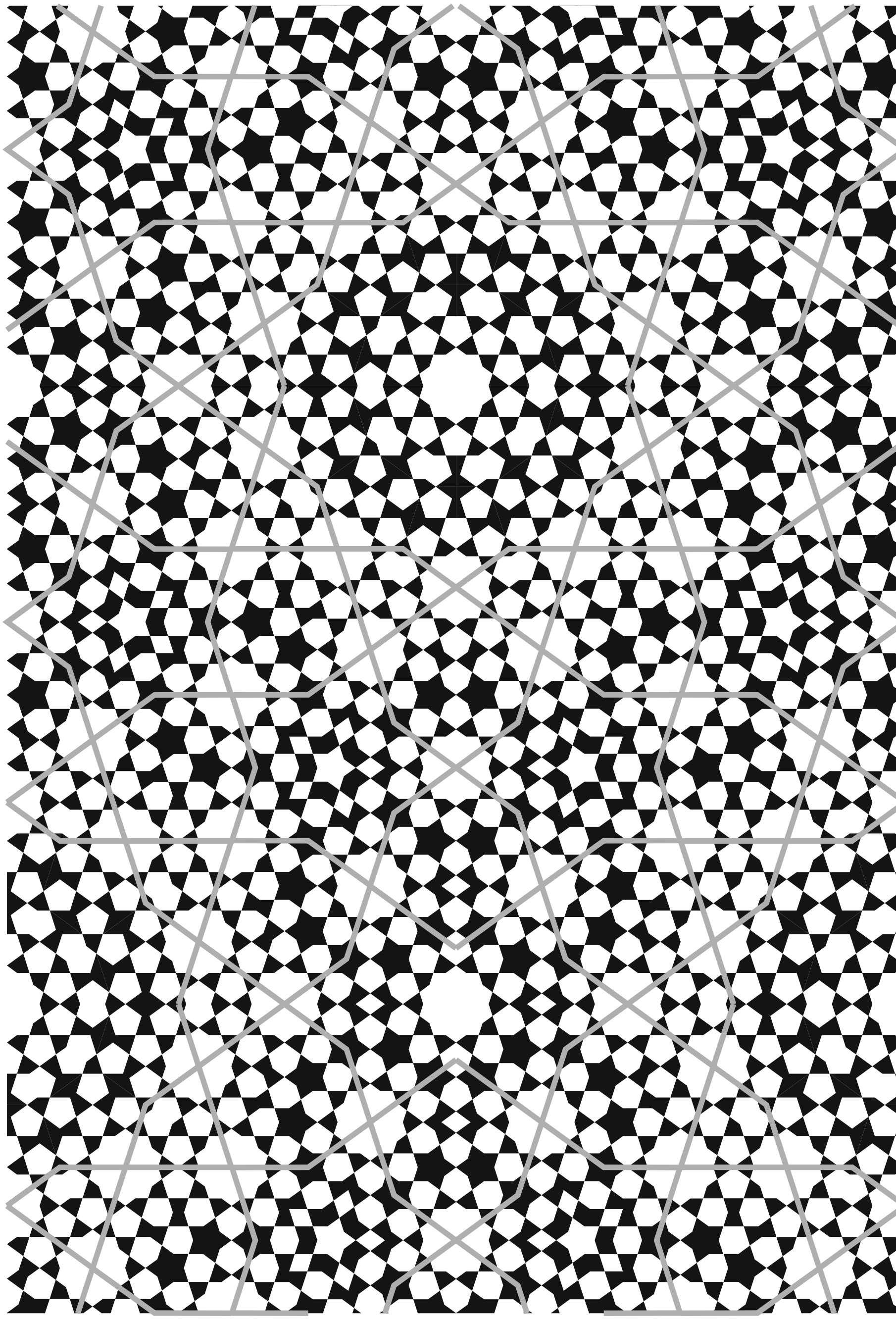


Figure 16. A 2-level design based on panel 32 of the Topkapi Scroll.

The bow-tie is notable by its absence from Figure 15. It suffers the same fate as the pentagon: the tiles at its two ends are forced and its waist cannot be tiled. (The large-scale polygonal network underlying panel 29 of the scroll has a quarter of a bow-tie in the top right corner surrounded by pieces of decagons, but it is not based on subdivision in the same way as the others.)

In Figure 16 the visible section of the large-scale design can also be found as a configuration in the small-scale design. However, larger sections reveal that the pattern is not scale invariant. This is a general limitation of these subdivisions. It is not possible to use the subdivisions of Figure 15 as the basis of a substitution tiling because, without subdivisions of the pentagon and bow-tie, the inflation process cannot be iterated.

A Design from the Alhambra

The design illustrated in Figure 18 forms the major part of a large panel in the Museum of the Alhambra—see [24] for a photograph. The panel has been assembled from fragments uncovered in 1958, but the original would have been from the 14th century. The lower part of the figure shows the finished design and the upper part shows a polygonal network that I propose as the underlying framework. The principal compositional element of the framework is a decagon surrounded by ten pentagons, which gives rise to

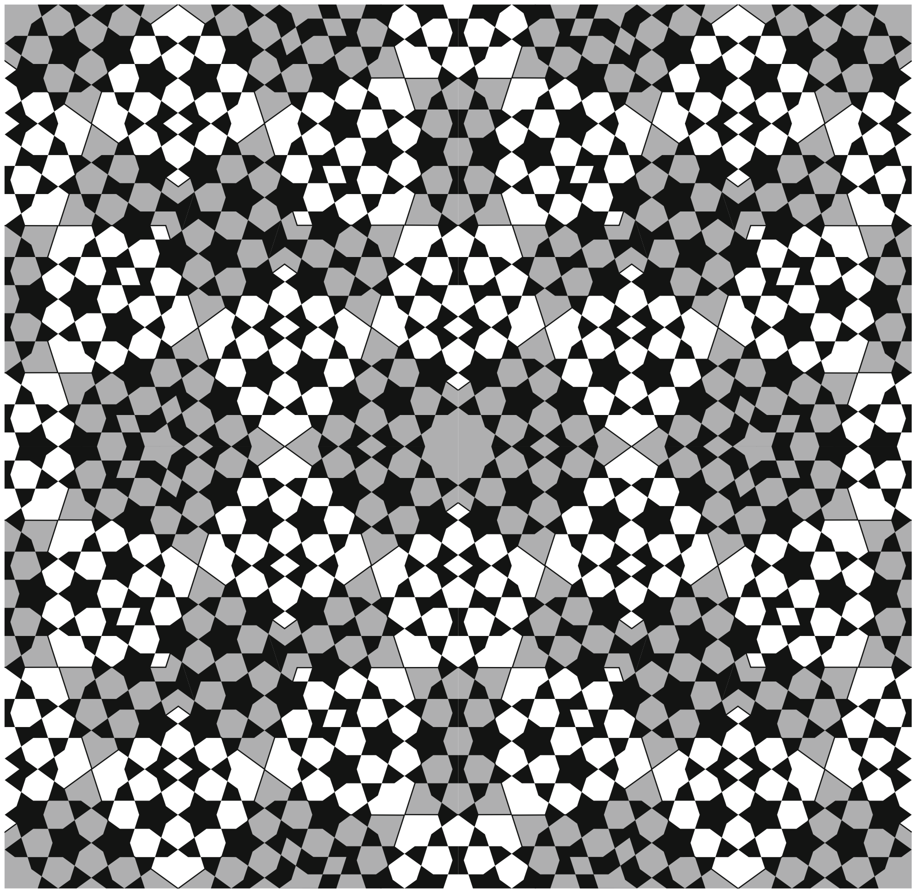


Figure 17. A 2-level design based on panel 34 of the Topkapi Scroll.

the 10-fold rose recurring as a leitmotif in the final design. Copies of this element are placed in two rings, visible in the top left of the figure—an inner ring of ten and an outer ring of twenty; adjacent elements share two pentagons. The connections between the inner and outer rings are of two kinds. The shaded rhombi contain the translation unit from the familiar periodic design of Figure 2(b). The construction of the design in the remaining spaces is shown in Figure 19: in part (b) the design is seen to be a subset of the configuration of pentagonal motifs of part (a), whereas (c) shows the same design over a network that includes half-barrels and one-tenth decagons—the polygons used in Figure 18. The edges in the resulting polygonal network are of two lengths, which are related as the side and diagonal of a pentagonal tile. The final design can be generated from this network using a generalisation of the PIC method: the short edges have incidence angle 72° and the long edges have incidence angle 36°. A 20-fold rose is placed in the centre; the tips of alternate petals meet 10-fold roses, and lines forming the tips of the intermediate petals are extended until they meet other lines in the pattern. The reconstructed rectangular panel also has quadrants of 20-fold roses placed in the four corners, a common feature of such panels that reflects the fact that most are subsets of periodic patterns. However, the quadrants are misaligned and are also the most heavily restored areas of the panel. I have omitted them from the figure.

This design is unusual in the large number of straight lines it contains that run across the figure almost uninterrupted. The marks in the bottom right corner of Figure 18 indicate the heights of horizontal lines; there are five families of parallels separated by angles of 36°. In some quasi-periodic tilings it is possible to decorate the prototiles with line segments that join up across the edges of the tiling to produce a grid of continuous straight lines that extend over the whole plane. These lines are called *Ammann bars*. The intervals between consecutive parallel Ammann bars come in two sizes, traditionally denoted by S and L (short and long). They form an irregular sequence that does not repeat itself and never contains two adjacent Ss or three adjacent Ls.

The lines in Figure 18 are not genuine Amman bars. Those marked with an asterisk do not align properly across the full width of the piece shown but deviate so that the S and L intervals switch sides. (Structural defects of this kind have been observed in quasicrystals, where they are known as phasons). The periodic design in Figure 2(b) has similar lines but its sequences repeat: the vertical 'Ammann bars' give sequence SLSL, the lines 36° from vertical give SLLSLL, and those 72° from vertical are not properly aligned.

Makovicky *et al* [24] propose Figure 18 as an example of a quasi-periodic design. They try to find a structural connection between it and the cartwheel element of the

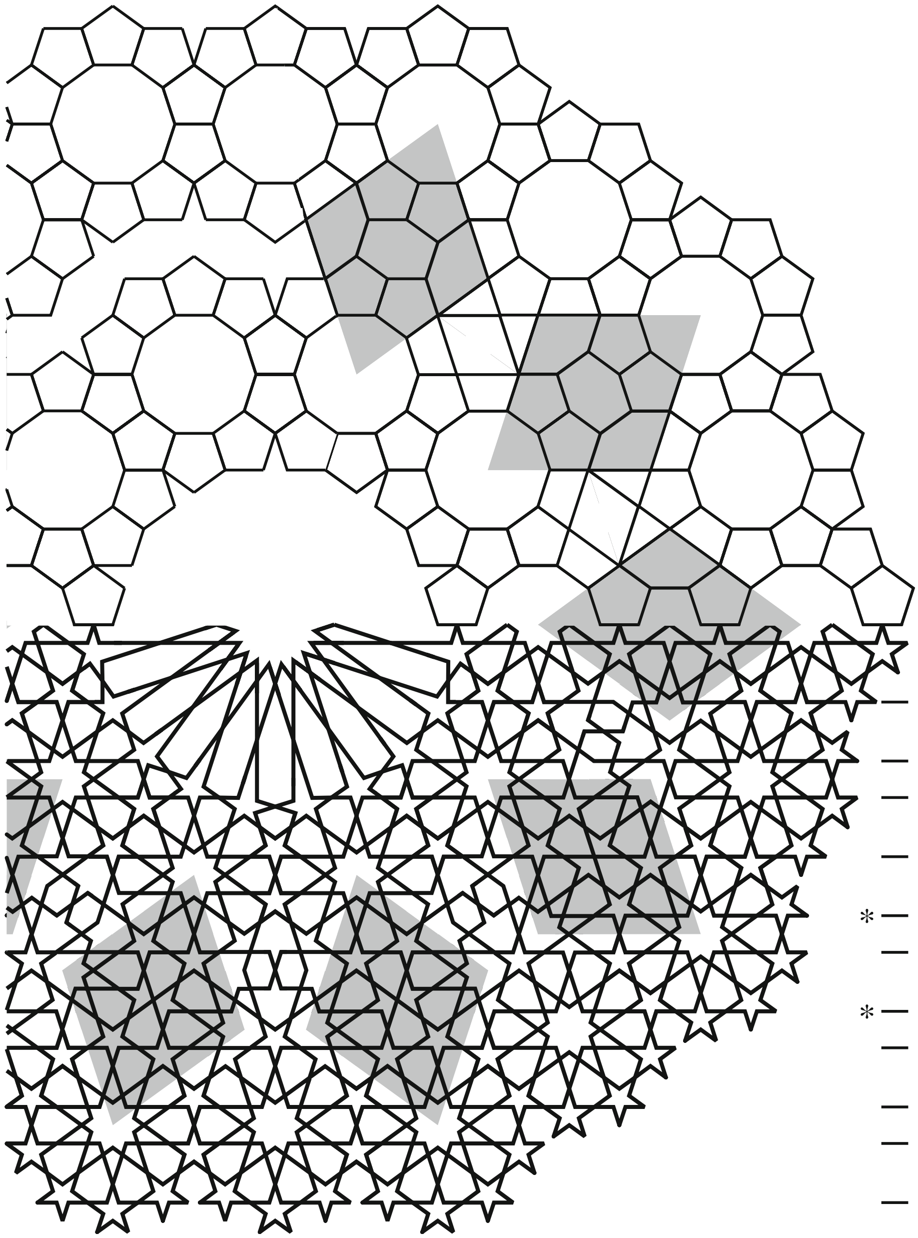
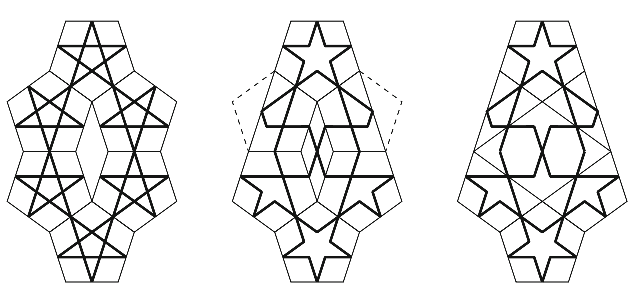


Figure 18. Construction of panel 4584 in the Museum of the Alhambra.



(a) (b) (c)

Figure 19.

Penrose tiling. After acknowledging that attempts to match kites and darts are problematic, they try to match it with a variant of the Penrose tiles, one discovered by Makovicky [20] as he studied the Maragha pattern shown in Figure 9(b). Their boldest assertion is Conclusion 6 [24, p. 125]:

> The non-periodic cartwheel decagonal pattern from the excavations in the Alhambra and from the Moroccan localities is based on a modified Penrose non-periodic tiling derived recently as 'PM1 tiling' by Makovicky… We conclude that a symmetrized PM1-like variety of Penrose tiling must have been known to the Merinid and Nasrid artesans (mathematicians) and was undoubtedly contained in their more advanced pattern collections.

Elsewhere in the paper, the authors are more cautious and realistic about the nature of their speculation. They offer an alternative construction based on an underlying radially symmetric network of rhombi whose vertices lie in the centres of the decagonal tiles [24, Fig. 23].

In order to classify a pattern as periodic or radially symmetric, we must have a large enough sample to be able to identify a template and the rules for its repetition. Similarly, to classify a pattern as quasi-periodic, we must describe a constructive process that allows us to see the given patch as part of a quasi-periodic structure covering the whole plane. It is not sufficient that geometric features of a design, such as rotation centres, can be shown to align with those of a familiar quasi-periodic tiling within a finite fragment. We need to find a procedure built on elements of the design. The set of tiles underlying Figure 18 and the set P_2 shown in Figure 13 are both large patches with 10-fold symmetry, but only in the second case do we know how to extend it quasi-periodically.

In my opinion, the design strategy underlying the Alhambra pattern does not require an understanding of Penrose-type tilings, and is based on little more than the desire to place large symmetric motifs (roses) in a radially symmetric pattern and fill the gaps. The construction outlined at the start of this section produces the complete design using methods and motifs believed to have been used by Islamic artists. The general structure has the same feel as Figure 5. The 'Ammann bars' are an artifact of the construction, although the structure of the design may have evolved and been selected to enhance their effect. They would also have helped to maintain accurate alignment of elements during its construction.

Designs from Isfahan

Figure 20 shows a 2-level design that, like the Topkapi Scroll examples above, is based on subdivision. The large-scale design is the stars and kites pattern derived from the bow-tie and decagon tiling of Figure 1(b). The subdivisions of the bow-tie and decagon used to generate the small-scale design are shown in Figures 21(a) and (c) with the large-scale pattern added in grey. The side of a composite tile is formed from the diagonals of two bobbins and one decagon. The pattern cannot be scale invariant: the polygonal network for the large-scale design contains a bow-tie surrounded by four decagons but this local arrangement does not occur in the small-scale network.

These subdivisions were derived by Lu and Steinhardt [17] from three hierarchical designs found on buildings in Isfahan. The grey areas in Figure 22 mark out the sections of the large-scale polygonal network underlying these designs: the rectangular strip runs around the inside of a portal in the Friday Mosque, the triangular section is one of a pair of mirror-image spandrels from the Darb-i Imam (shrine of the Imams), and the arch is a tympanum from a portal, also from the Darb-i Imam—see [17, 35] for photographs. Bonner [2] gives an alternative subdivision scheme for the Darb-i Imam arch using the tiling of Figure 2(a) as the basis for the large-scale design.

The mosaic in the Darb-i Imam tympanum differs from the symmetrically perfect construction of Figure 20 in several places. For example a bow-tie/bobbin combination like Figure 7(a) in the top right corner of the central composite bow-tie is flipped; bow-tie/bobbin combinations in the corners of the upper composite decagon are also flipped; a decagon at the lower end of the curved section of the boundary on each side is replaced by Figure 7(d). The modifications to the composite decagon appear to be deliberate as the same change is applied uniformly in all corners. Replacing the small decagons may make it easier to fit the mosaic into its alcove. The bow-tie anomaly is possibly a mistake by the craftsman.

If we want to use the Isfahan subdivisions as the basis of a substitution tiling, we need to construct a companion subdivision of the bobbin tile. In doing so we should

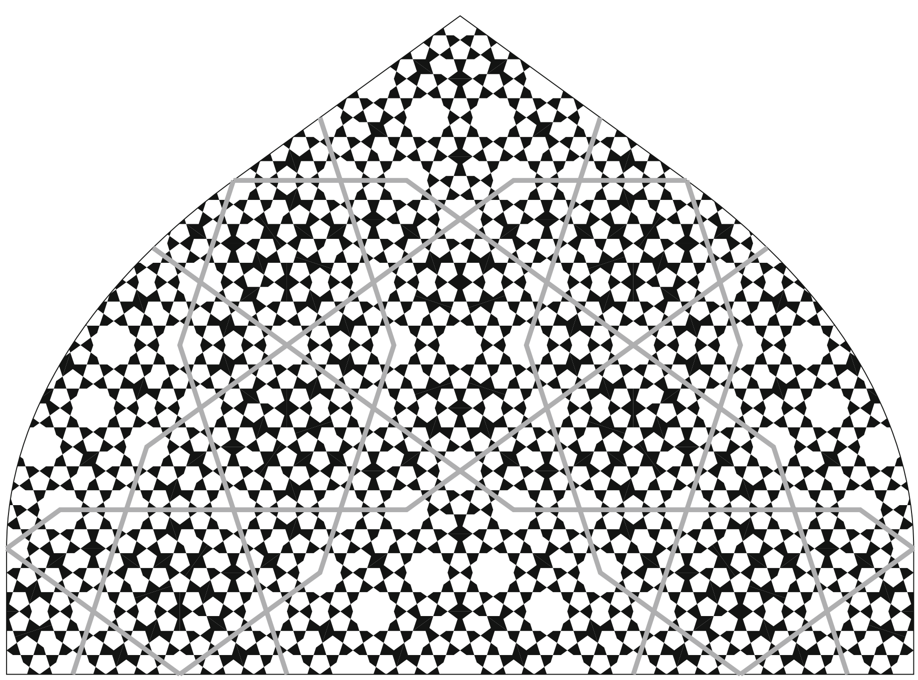


Figure 20. A 2-level design modelled on the Darb-i Imam, Isfahan.

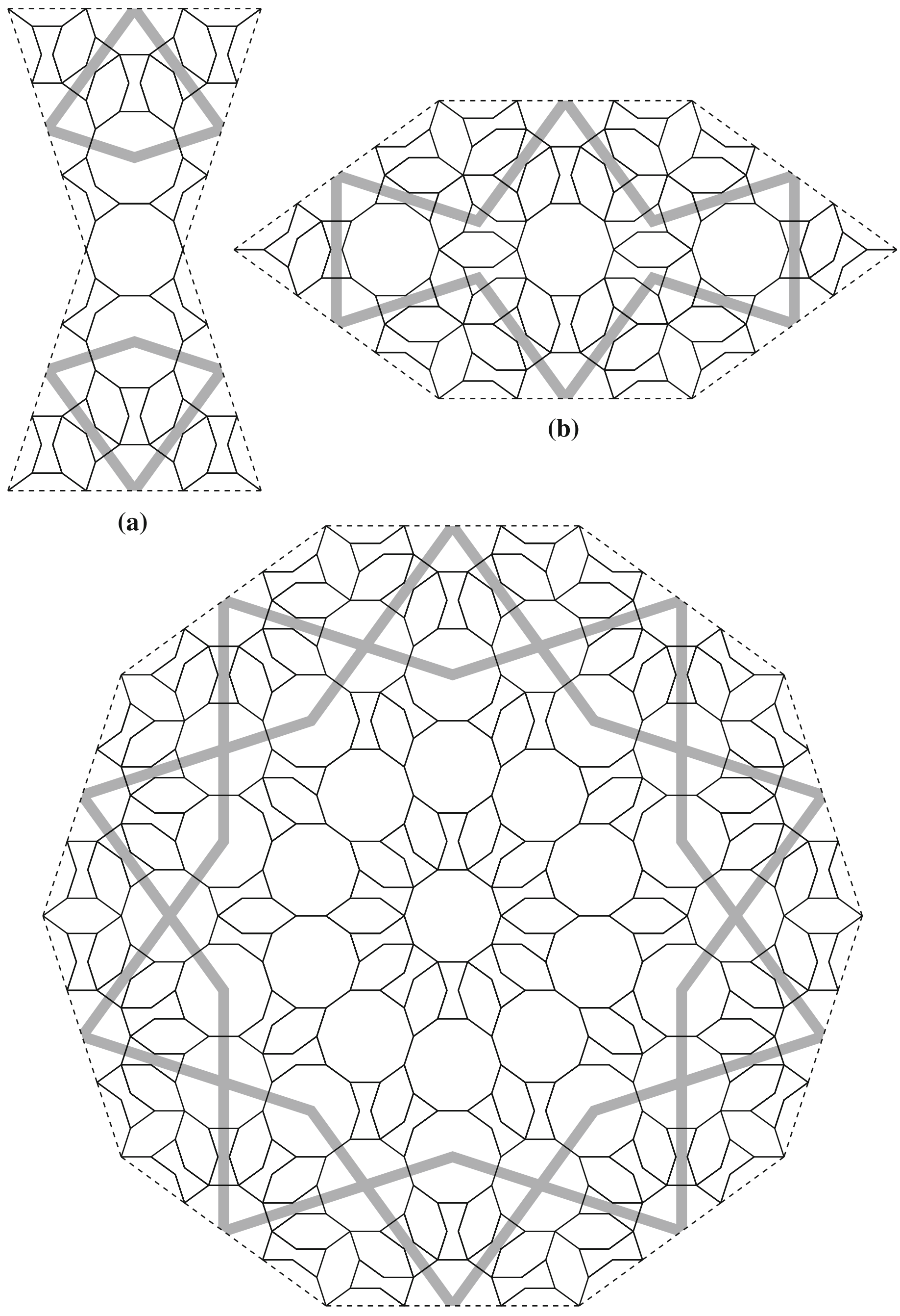


Figure 21. Subdivisions (a) and (c) are derived from designs on buildings in Isfahan [17]. The scale factor is $4 + 2\sqrt{5} \approx 8.472$.

emulate the characteristics of the two samples—properties such as the mirror symmetry of the subdivisions, and the positions of the tiles in relation to the grey lines. Notice that focal points such as corners or intersections of the grey lines are always located in the centres of decagons, and the interconnecting paths pass lengthwise through bow-ties. Figure 21(b) shows my solution: it satisfies some of these criteria, but it is spoilt by the fact that some of the corners of the grey lines are so close together that decagons centred on them overlap, and there is a conflict between running the path through a bow-tie and achieving mirror symmetry at the two extremes. This extra subdivision enables the inflation process to be performed, but the resulting tilings are probably of mathematical interest only. The large scale factor for the subdivisions yields a correspondingly large growth rate for the inflation. After two inflations of a decagon the patch would contain about 15000 tiles; for comparison, the patch shown in Figure 13 contains about 1500 tiles.

Lu and Steinhardt use the Isfahan patterns in their discussion of quasi-periodicity. Commenting on the spandrel, they say [17, p. 1108]:

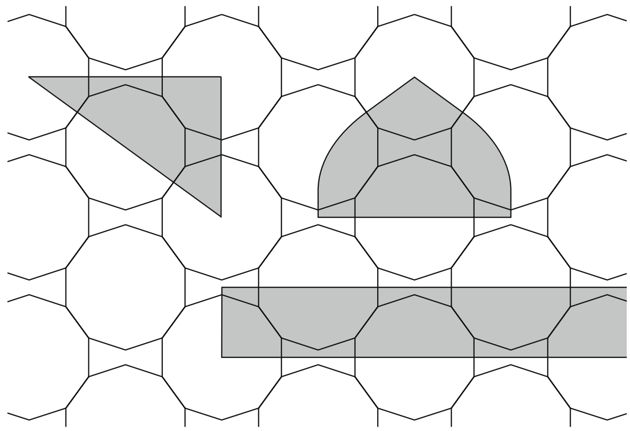


Figure 22. Sections of the bow-tie and decagon tiling used in the Isfahan patterns.

> The Darb-i Imam tessellation is not embedded in a periodic framework and can, in principle, be extended into an infinite quasiperiodic pattern.

By this they mean that the visible fragment of the large-scale design is small enough that no translational symmetry is immediately apparent and so the patch could be part of a non-periodic tiling. If we only have access to a finite piece of any tiling, it is impossible to decide whether it is periodic without further information on its local or global structure. Although the lack of conspicuous periodicity in the Darb-i Imam design could be interpreted as a calculated display of ambiguity on the part of the artist, to me it seems more likely to be the result of choices influenced by aesthetic qualities of the design, and the relative sizes of the tesserae in the small-scale pattern and the area to be filled. The fact that the same periodic tiling is a basis for all three Isfahan designs makes it a good candidate for the underlying organising principle. Translation in one direction is visible in the Friday Mosque pattern.

Lu and Steinhardt also observe that the medieval artists did not subdivide a single large tile but instead used a patch containing a few large tiles arranged in a configuration that does not appear in the small-scale network. They then remark [17, p. 1108]:

> This arbitrary and unnecessary choice means that, strictly speaking, the tiling is not self-similar, although repeated application of the subdivision rule would nonetheless lead to [a non-periodic tiling].

This gives the impression that, if the medieval craftsmen had wanted to, they could have started with a single tile and inflated it until it covered the available space. But we must beware of seeing modern abstractions in earlier work. There is no evidence that medieval craftsmen understood the process of inflation. The mosaics require only one level of subdivision, and they do not contain a subdivision of the bobbin that would be needed to iterate the inflation.

In my opinion the Isfahan patterns, like the 2-level designs in the Topkapi Scroll, are best explained as an application of subdivision to generate a small-scale filling of a periodic large-scale design. Furthermore, the choice of the large-scale design seems far from arbitrary: it is one of the oldest and most ubiquitous decagonal star patterns, and as such it would have been very familiar to medieval viewers and recognised even from a small section.

Connections with Penrose Tilings

The use of subdivision and inflation to produce quasi-periodic tilings with forbidden rotation centres came to prominence in the 1970s with investigations following the discovery of small aperiodic sets of tiles, the Penrose kite and dart being the most famous example. Penrose tilings have local 5-fold and 10-fold rotation centres and the fact that some Islamic designs share these unusual symmetry properties has prompted several people to explore the connections between the two [1, 17, 20, 24, 27].

Figure 23 shows subdivisions of the kite and dart into the bow-tie, bobbin, and decagon tiles. As in earlier examples, the sides of the kite and dart lie on mirror lines of the tiles. Using this substitution, any Penrose tiling can be converted into a design in the Islamic style [27]. Furthermore, because the kite and dart are an aperiodic set, such a design will be non-periodic.

The transition can also proceed in the other direction. Figure 24 shows subdivisions of the three Islamic tiles into kites and darts. Two of the patches are familiar to students of Penrose tilings: (a) is the long bow-tie component of Conway worms and (b) is the hub of the cartwheel tiling. Notice also that (b) is assembled from (a) and (c) in the manner of Figure 7(d).

Kites and darts come with matching rules to prohibit the construction of periodic tilings when the tiles are assembled like a jigsaw. In Figure 24 the two corners at the 'wings' of each dart and the two corners on the mirror line of each kite are decorated with grey sectors; the matching rule is that grey corners may only be placed next to other grey corners. This prevents, for example, the bow-tie and the decagon in the figure from being assembled in the stars and kites pattern: it is not possible to place two bow-ties on opposite corners of a decagon.

The markings on the kites and darts in Figure 24 endow the composite tiles with a matching rule of their own. Each side of a composite tile has a single grey spot that divides its length in the golden ratio; we decorate each side with an arrow pointing towards the short section. Instead of defining the matching rule at the vertices of the tiling, as

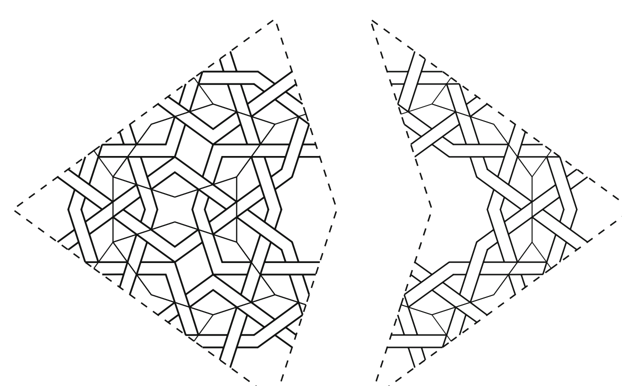


Figure 23. Subdivisions of the Penrose kite and dart.

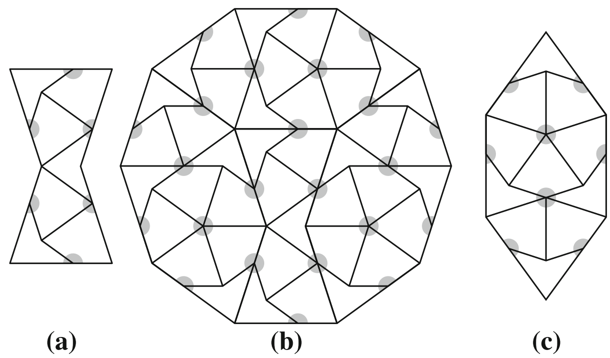


Figure 24. Patches of Penrose kites and darts.

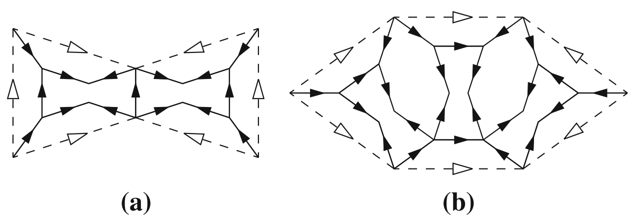


Figure 25. Subdivisions of marked tiles that preserve the markings. The scale factor is $\frac{1}{2}\left(3+\sqrt{5}\right) \approx 2.618$.

with the Penrose example previously described, we place constraints on the edges of the tiling: the arrows on the two sides forming an edge of the tiling must point in the same direction. With these markings and matching rule, the bow-tie and bobbin are an aperiodic set. To prove this note that the subdivisions in Figure 25 show that we can tile the plane by inflation, and that any periodic tiling by bow-ties and bobbins could be converted into a periodic tiling by kites and darts but this is impossible. The substitution matrix for these marked tiles is associated with the Fibonacci sequence and the ratio of bobbins to bow-ties in a substitution tiling is the golden ratio. Notice that a horizontal line running through the centre of a composite bow-tie passes lengthwise through the small bow-ties and short-ways across the small bobbins. Inflation produces a longer line with the same properties, and a substitution tiling will contain arbitrarily long such lines. Any infinite lines must be parallel as they cannot cross each other. These lines inherit their own 1-dimensional substitution rule.

Conclusions

In the preceding sections I have described methods for constructing Islamic geometric patterns, given a brief introduction to the modern mathematics of substitution tilings, and analysed some traditional Islamic designs. The conclusions I reached during the course of the discussion are isolated and summarised here:

1. It is possible to construct quasi-periodic tilings from the set of prototiles used by Islamic artists (Figure 6). Examples can be generated as substitution tilings based on inflation or using a matching rule with marked versions of the tiles.
2. Islamic artists did use subdivision to produce hierarchical designs. There are examples illustrating the method in the Topkapi Scroll, and three designs on buildings in Isfahan can be explained using this technique. Indeed, their prototiles are remarkable in their capacity to form subdivisions of themselves in so many ways.
3. There is no evidence that the Islamic artists iterated the subdivision process—all the designs I am aware of have only two levels. This is to some degree a practical issue: the scale factor between the small-scale and large-scale designs is usually large and the area of the design comparatively small. With the subdivisions used in the Topkapi Scroll, iteration is impossible as composite versions of the pentagon and bow-tie do not exist.
4. There is no evidence that the Islamic artists used matching rules. Ammann bars are the nearest thing to a form of decoration that could have been used to enforce non-periodicity. Similar lines that appear on some designs are a by-product of the construction, not an input to the design process, although the designs may have been selected because this feature was found attractive.
5. The designs analysed in this article do not provide evidence that Islamic artists were aware of a process that can produce quasi-periodic designs. They are periodic, generated by reflections in the sides of a rectangle, or are large designs with radial symmetry. The multi-level designs are hierarchical, not scale invariant.

In this article I have concentrated on designs with local 5-fold symmetry. In Spain and Morocco there are analogous designs with local 8-fold symmetry, including some fine 2-level designs in the Patio de las Doncellas in the Alcazar, Seville—see [22] for photographs. The geometry of the polygonal networks underlying these designs is grounded on the $\sqrt{2}$ system of proportions rather than the golden ratio. Plans of muqarnas (corbelled ceilings built by stacking units in tiers and progressively reducing the size of the central hole to produce a stalactite-like dome) sometimes display similar features. These networks have a strong resemblance to the Ammann–Beenker quasi-periodic tiling composed of squares and 45°–135° rhombi [33]. This tiling is another substitution tiling that can be generated by subdivision and inflation; the tiles can also be decorated with line segments to produce Ammann bars. Similar claims to those assessed in this article have been made for some of the Islamic 8-fold designs [2, 6, 22, 23].

To me, it seems most likely that the Islamic interest in subdivision was for the production of multi-level designs. Islamic artists were certainly familiar with generating designs by applying reflection, rotation and translation to repeat a template. They probably had an intuitive understanding of the crystallographic restriction and a feeling that global 5-fold and 10-fold rotation centres are somehow incompatible with periodicity. They did have the tools available to construct quasi-periodic designs but not the theoretical framework to appreciate the possibility or significance of doing so.

ACKNOWLEDGMENTS

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